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ON THE INITIAL VALUE PROBLEM ASSOCIATED TO A GENERALIZATION OF THE REGULARIZED BENJAMIN-ONO EQUATION

JOHN BOLANOS & GUILLERMO RODRÍGUEZ-BLANCO

Department of Mathematics, National University of Colombia, Bogota, Colombia

ABSTRACT

Our aim is to establish local and global well-posedness results in weighted Sobolev spaces $\mathcal{F}_{s,r}(\mathbb{R})$ via contraction principle and prove a unique continuation property for a generalization of the regularized Benjamin-Ono Equation.

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Original Article

1. INTRODUCTION

Many phenomena that occur in Physics and Engineering are modeled by partial differential equations. Some kind of them, some very remarkable, such like the KdV equation, B-O equation, Schrödinger equation, to mention some of them, are the partial differential equations of evolution, whose name is due to the fact that one of the independent variables is time. These models are also important in Mathematics, since they lead to problems such as, local and global well-posedness, stability of solitary waves, principles of single continuation, propagation of regularity, to mention some of them; whose solutions rescue classic techniques of Analysis, as they also give rise to new ideas that have led to their solution. In this work, we will deal with the initial value problem associated with a generalization of the regularized Benjamin-Ono equation (gr-BO). More accurately, We consider the problem:

$$\begin{cases} u_t + D^{1+\alpha}u_t + u_x + uu_x = 0, \quad t,x \in \mathbb{R},\ 0<\alpha<1, \\ u(0,x) = \varphi(x) \in H^s(\mathbb{R}), \end{cases} \tag{1}$$

where, $D^s f = (|\xi|^s \hat{f})^\vee$, is the homogeneous derivative of order $s \in \mathbb{R}$; $\hat{}$ is the Fourier transform and $\vee$ its inverse.

When $\alpha = 0$, equation (1) is the well-known regularized Benjamin-Ono equation (r-BO). Results for this equation about well-posedness in Sobolev spaces $H^s(\mathbb{R})$, for $s > 1/2$ and weighted Sobolev spaces $\mathcal{F}_{s,r}(\mathbb{R})$ for $s > 1/2$, $0 \le r < 5/2$ together with unique continuation principles were obtained by Germán Fonseca, Guillermo Rodríguez-Blanco and Wilson Sandoval in [5]. Following the ideas of this work, we obtained similar results for (1), and in order to make the reading of this work more enjoyable, we present a series of preliminaries in the following section.

2. PRELIMINARIES

Theorem 1: If $0 < b < 1$ and $1 < p < \infty$, then

$$\|D^b(fg)\|_{L^p} \leq C(\|g\|_\infty \|D^b f\|_{L^p} + \|f\|_\infty \|D^b g\|_{L^p}) \tag{2}$$

Proof. See [7].

Definition 2: *Let $b \in (0,1)$ and f measurable on $\mathbb{R}^n$ with complex values. The Stein derivative is defined by:*

$$\mathcal{D}^b f(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}} dy\right)^{1/2} \tag{3}$$

Definition 3: *For $s \in \mathbb{R}$. We note by $L_s^p(\mathbb{R}^n)$ the space of all functions $f \in L^p(\mathbb{R}^n)$ such that $(1-\Delta)^{s/2} f \in L^p(\mathbb{R}^n)$. The norm in this space is given by*

$$\|f\|_{s,p} = \left\|(1-\Delta)^{\frac{s}{2}} f\right\|_{L^p}$$

for each f on $L^p(\mathbb{R}^n)$

The following theorems characterize the spaces $L_s^p(\mathbb{R}^n)$ in terms of the Stein derivative:

Theorem 4: *Let $b \in (0,1)$ and $2/n + 2b \leq p < \infty$. Then, $f \in L_b^p(\mathbb{R}^n)$ if and only if*

1. $f \in L^p(\mathbb{R}^n)$.

2. $\mathcal{D}^b f(x) \in L^p(\mathbb{R}^n)$.

with,

$$\|f\|_{b,p} = \left\|(1-\Delta)^{\frac{b}{2}}\right\|_{L^p} \cong \|f\|_{L^p} + \|D^b f\|_{L^p} \cong \|f\|_{L^p} + \|\mathcal{D}^b f\|_{L^p}$$

Proof. See [9] or [10]

Theorem 5: *Let $b \in (0,1)$ and $1 \leq p < \infty$. If $f, g: \mathbb{R}^n \to \mathbb{C}$ are measurable functions, then*

$$\|\mathcal{D}^b(fg)\|_{L^2} \leq \|f\mathcal{D}^b g\|_{L^2} + \|g\mathcal{D}^b f\|_{L^2} \tag{4}$$

Proof. See Proposition 1 in [8]

Proposition 6: *For $b \in (0,1)$*

$$\|\mathcal{D}^b f\|_\infty \leq c_b(\|f\|_\infty + \|\partial_x f\|_\infty) \tag{5}$$

Proof. It is a direct consequence of the definition of the Stein derivative (3)

Proposition 7: *Let $P(x)$ and $Q(x)$ polynomials of degree m and n respectively with $0 < m < n$, for $b \in (0,1)$, $\varphi \in L^2(\mathbb{R})$ and $\frac{P}{Q} \in C(\mathbb{R})$ then*

$$\left\|D_x^b\left(\frac{P}{Q} \cdot \varphi\right)\right\|_0 \leq c(\|\varphi\|_0 + \|\mathcal{D}_x^b \varphi\|_0)$$

Proof. See Proposition 2.11 in [3]

The following lemmas are important to obtain results for (1) on Sobolev spaces with fractional weights, in which the Stein derivative plays a leading role.

Lemma 8: *Let* $b \in (0,1)$, $\alpha > 0$. *Then,*

$\mathcal{D}_x^b\left(e^{\frac{-itx}{1+|x|^{1+\alpha}}}\right) \le C(\alpha, b)t^b$(6)

for all $t > 0$.

Proof. See proposition 2.13 in [3] **Proposition 9** *Let* $\chi \in C_0^\infty$, *a function such taht supp* $\chi \subseteq [-2,2]$ *and* $\chi \equiv 1$ *in* $(-1,1)$. *For any* $b \in (0,1)$ *and* $\theta > 0$,

$$\mathcal{D}^b\left(|\xi|^\theta \chi(\xi)\right)(\eta) \sim \begin{cases} c|\eta|^{\theta-b} + c_1, & \theta \neq b, \quad |\eta| \ll 1, \\ c(-\ln|\eta|)^{\frac{1}{2}}, & \theta = b, \quad |\eta| \ll 1, \\ \frac{c}{|\eta|^{\frac{1}{2}+b}}, & \quad |\eta| \gg 1, \end{cases}$$

with $\mathcal{D}^b(|\xi|^\theta \chi(\xi))(\cdot)$ continous in $\eta \in \mathbb{R} - \{0\}$. In particular, one has that

$$\mathcal{D}^b(\xi|^\theta \chi(\xi)) \in L^2(\mathbb{R}) \text{ if and only if } b < \theta + 1/2. \tag{7}$$

Similar result holds for $\mathcal{D}^b(|\xi|^\theta \mathrm{sgn}(\xi)\chi(\xi))$.

Proof. See proposition 2.9 in [4].

Proposition 10 *If* $\hat{f}(0) = 0$ *and* $0 < \alpha < 1$,

$$\left\|D_\xi^\alpha\left(\mathrm{sgn}(\xi)\hat{f}\right)\right\|_0 \le \left\|\hat{f}\right\|_0 + \left\|D_\xi \hat{f}\right\|_0$$

Proof. See proposition 2.19 in [3]

3. LOCAL WELL-POSEDNESS ON $H^s(\mathbb{R})$

For convenience we will write the initial value problem (1) in the following form:

$$\begin{cases} u_t = Au + f(u) \\ u(0) = \varphi \in H^s(\mathbb{R}), \end{cases} \tag{8}$$

which is equivalent to the **Integral Equation**,

$$u(t) = E(t)\varphi + \int_0^t E(t-\tau) f(u(\tau))d\tau, \tag{9}$$

where,

$$A = -\partial_x(1 + D_x^{1+\alpha})^{-1},\ f(u) = A(u^2)$$

$$E(t)\varphi = e^{At}\varphi = \left(e^{\frac{-i\xi}{1+|\xi|^{1+\alpha}}t}\hat{\varphi}\right)^\vee$$

Proposition 11: *The operator* $A = -\partial_x(1 + D_x^{1+\alpha})^{-1}$ *is bounded on* $H^s(\mathbb{R})$, *if* $\alpha \ge 0$.

Proof. Let $\varphi \in H^s(\mathbb{R}^2)$

$$\|A(\varphi)\|_s^2 = \|-\partial_x(1+D_x^{1+\alpha})^{-1}\varphi\|_s^2$$

$$= \int_{R^2} (1+\xi^2)^s \left|\frac{i\xi}{1+|\xi|^{1+\alpha}}\hat{\varphi}\right|^2 d\xi \quad \text{if } \alpha \geq 0 \text{ we have}$$

$$\leq \int_{R^2} (1+\xi^2)^s |\hat{\varphi}|^2 d\xi = \|\varphi\|_s^2$$

Proposition 12: *The application* $E: \mathbb{R} \mapsto \mathfrak{B}(H^s(\mathbb{R}))$ *defined by* $E(t)\varphi = e^{At}\varphi$ *for* $\varphi \in H^s(\mathbb{R})$ *and* $t \in [0,\infty)$ *is a strongly continuous unitary group.*

Proof. The proof is simple and direct, for that reason we do not present it here.

Proposition 13: *If* $s > 1/2$ *and* $\alpha \geq 0$, *the fucntion* $f(u) = A(u^2)$ *satisfies the condition of local lipschitz, i.e,*

$$\|f(u) - f(v)\|_s \leq L(\|u\|_s, \|v\|_s)\|u - v\|_s, \tag{10}$$

for all $u, v \in H^s(\mathbb{R})$, where $L(\|u\|_s, \|v\|_s) = \|u\|_s + \|v\|_s$

Proof. Since $H^s(\mathbb{R})$ is a Banach algebra for $s > 1/2$, the proof is a consequence of this fact and the proposition (11).

To proof the existence of a solution of (8) we consider the set:

$$\mathfrak{X}_s(T, M, \varphi) = \{u \in C([0,T]; H^s(\mathbb{R}));\ \|u(t) - E(t)\varphi\|_s \leq M\}, \tag{11}$$

which is a complete metric subspace of $C([0,T]; H^s)$ with the metric

$$d_s(v, w) = \sup_{t\in[0,T]} \| v(t) - w(t) \|_s = \| v - w \|_{s,\infty}$$

and the application

$$\Psi(v)(t) = E(t)\varphi + \int_0^t E(t-\tau) f\big(v(\tau)\big) d\tau$$

Proposition 14: *If* $\alpha \geq 0$ *and* $s > 1/2$, *then the application* Ψ *satisfy:*

1. Exists $T_1(\|\varphi\|_s, M) \geq 0$ such that $\Psi(v)(t) \in \mathfrak{X}_s(T_1, M, \varphi)$

2. Exists $T_2(\|\varphi\|_s, M) \geq 0$ such that $\Psi(v)(t)$ is a contraction.

Proof. Part **1**:

$$\|\Psi(v)(t) - E(t)\varphi\|_s \leq \int_0^t \left\|E(t-\tau) f\big(v(\tau)\big)\right\|_s d\tau$$

$$\leq C_s \int_0^t \left\|\big(v(\tau)\big)\right\|_s^2 d\tau$$

$$\leq C_s(M + \|\varphi\|_s)^2 t \tag{12}$$

choosing $T1 > 0$ such that the right side of 12 is less than M we get the result.

Part **2**:

$$\|\Psi(v)(t) - \Psi(w)(t)\|_s \leq \int_0^t \left\|E(t-\tau)\Big(f\big(v(\tau)\big) - f\big(w(\tau)\big)\Big)\right\|_s d\tau$$

$$\leq \int_0^t \left\| \left(f\left(v(\tau)\right) - f\left(w(\tau)\right) \right) \right\|_s d\tau \text{by}(10)$$

$$\leq C_s \int_0^t \left(\|v(\tau)\|_s + \|w(\tau)\|_s\right)\|v(\tau) - w(\tau)\|_s d\tau$$

$$\leq 2C_s(M + \|\varphi\|_s) \int_0^t \|v(\tau) - w(\tau)\|_s d\tau$$

$$\leq C_s(M + \|\varphi\|_s)\|v - w\|_{s,\infty} t$$

Choosing $T_2 > 0$ such that $C_s(M + \|\varphi\|_s)T_2 < 1$ we get that Ψ is a contraction.

The above proposition together with Banach's fixed point theorem implies the following theorem.

Theorem 15: *If* $\varphi \in H^s(\mathbb{R}^2)$, $s > 1/2$ *and* $\alpha \geq 0$, *exists* $T = T(\|\varphi\|_s, M) > 0$ *and* $u \in C([0,T]; H^s(\mathbb{R}^2))$ *that satisfies the integral equation (9).*

Uniqueness and continuous dependence are followed by standard methods.

Lemma 16 *Suppose that* $\varphi \in H^s$ *for* $s > 1/2$ *and let* $u \in C([0,T], H^s(\mathbb{R}))$ *be the solution of (1), then*

$\|u(t)\|_{\frac{1+\alpha}{2}} \sim \|\varphi\|_{\frac{1+\alpha}{2}}$.

Proof. The equation (1) implies that

$$(1 + D^{1+\alpha})u_t = -\partial_x u - \frac{1}{2}\partial_x u^2. \tag{13}$$

Now

$$\|u\|^2_{\frac{1+\alpha}{2}} \sim \langle J^{\frac{1}{2}}u, J^{\frac{1}{2}}u \rangle,$$

where, $J^s = (1 + D^{1+\alpha})^s$. Therefore, it easily follows that

$$\frac{d}{dt}\langle J^{\frac{1}{2}}u, J^{\frac{1}{2}}u\rangle = 2\langle (1 + D^{1+\alpha})^{\frac{1}{2}}u, (1 + D^{1+\alpha})^{\frac{1}{2}}((1 + D^{1+\alpha})^{-1}(-\partial_x u - \frac{1}{2}\partial_x u^2))\rangle$$

$$= 2\,\langle u, -\partial_x u - \frac{1}{2}\partial_x u^2\rangle$$

$$= -2\langle u, \partial_x u\rangle - \langle u, \partial_x u^2\rangle$$

$$= 0$$

This implies the result.

Next theorem shows that the I.V.P (1) is globally well-posed in $H^s(\mathbb{R})$, $s > 1/2$. The key point is to obtain *a priori* estimates of the Sobolev norm in $H^s(\mathbb{R})$, $s > 1/2$ with the help of the *a priori* bound of the $H^{\frac{1+\alpha}{2}}$ norm in Lemma 2.2 and the Kato-Ponce commutator, [6],

$$|fg|_{s,p} \leq c(\|f\|_{L^\infty}|g|_{s,p} + \|g\|_{L^\infty}|f|_{s,p}) \quad \text{for } s > 0,\ 1 < p < \infty,$$

and the Brezis-Gallouetâ€™s inequality, [2], which in dimension one is

$$\|f\|_{L^\infty} \leq c\left(1 + \sqrt{\log(1 + \|f\|_s)}\|f\|_{\frac{1}{2}}\right), \quad \text{for } s > \frac{1}{2}.$$

Theorem 17 *The Cauchy problem (1) is g.w.p. in* $H^s(\mathbb{R})$, $s > 1/2$.

Proof. Let $\varphi \in H^s((R))$, the integral equation 9 implies that

$$\|u(t)\|_s \le \|\varphi\|_s + \int_0^t \|u(\tau)\|_\infty \|u(\tau)\|_s d\tau$$

$$\le \|\varphi\|_s + c_0 \int_0^t \left(1 + \sqrt{\log(1 + \|u(\tau)\|_s)}\ \right)\|u(\tau)\|_s d\tau =: \Psi(t),$$

where c_0 only depends on $\|\varphi\|_{\frac{1+\alpha}{2}}$. The above lemma and this inequality imply that

$$\Psi'(t) = c_0\left(1 + \sqrt{\log(1 + \|u(\tau)\|_s)}\ \right)\|u(\tau)\|_s$$

$$\le c_0\left(1 + \sqrt{\log\big(1 + \Psi(t)\big)}\ \right)\Psi(t)$$

$$\le c_0\left(1 + \log\big(1 + \Psi(t)\big)\right)\Psi(t)$$

Then, there exists $c_1 > 0$ such that,

$$\frac{d}{dt}\log\left(1 + \log\big(1 + \Psi(t)\big)\right) \le c_1$$

and hence, there are constants $c_2 > 0$ and $c_3 > 0$ such that for every $t \in [0, T]$, $\|u(t)\|_s \le e^{c_2 e^{c_3 t}}$.

4. THE PROBLEM GR-BO ON WEIGHTED SPACES $\mathcal{F}_{s,r}$

In this section, we study the IVP (1), on weighted spaces $\mathcal{F}_{s,r}$, in which we establish local well-posedness and single continuation principles. To study the local well-posedness on these spaces we need to bound the operator

$A = -\partial_x(1 + D_x^{1+\alpha})^{-1}$ and the group $E(t)$ en $\mathcal{F}_{s,r}$.

Keeping this in mind, we calculate the first derivatives of the function $\frac{-i\xi}{1+|\xi|^{1+\alpha}}$.

$$\partial_\xi\left(\frac{-i\xi}{1+|\xi|^{1+\alpha}}\right) = \frac{-i(1-\alpha|\xi|^{1+\alpha})}{(1+|\xi|^{1+\alpha})^2}$$

$$\partial_\xi^2\left(\frac{-i\xi}{1+|\xi|^{1+\alpha}}\right) = \frac{-i\alpha(1+\alpha)|\xi|^{2\alpha+1}\operatorname{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3} + \frac{i(1+\alpha)(2+\alpha)|\xi|^{\alpha}\operatorname{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3}$$

$$\partial_\xi^3\left(\frac{-i\xi}{1+|\xi|^{1+\alpha}}\right) = \frac{i\alpha(1+\alpha)(2+\alpha)|\xi|^{1+3\alpha}}{(1+|\xi|^{1+\alpha})^4} - \frac{i(1+\alpha)(4\alpha^2+8\alpha+6)|\xi|^{2\alpha}}{(1+|\xi|^{1+\alpha})^4} + \frac{i\alpha(1+\alpha)(2+\alpha)|\xi|^{\alpha-1}}{(1+|\xi|^{1+\alpha})^4}$$

Proposition 18: The operator $A = -\partial_x(1 + D_x^{1+\alpha})^{-1}$ is bounded $\mathcal{F}_{s,r}(\mathbb{R})$ for $0 \le r < 5/2 + \alpha < 3$.

Proof.

$$\|A\varphi\|_{\mathcal{F}_{s,r}} = \|A\varphi\|_s + \|A\varphi\|_{L_r^2}$$

The first term is bounded in a similar way to the proposition 11. For the second term lets suppose that $r = 2$,

$$\|A\varphi\|_{L_2^2} = \left\|\partial_\xi^2\left(\frac{-i\xi}{1+|\xi|^{1+\alpha}}\hat{\varphi}\right)\right\|_0$$

$$\le \left\|\frac{\alpha(1+\alpha)|\xi|^{2\alpha+1}\operatorname{sgn}(\xi)i}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right\|_0 + \left\|\frac{(1+\alpha)(2+\alpha)|\xi|^{\alpha}\operatorname{sgn}(\xi)i}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right\|_0 + 2\left\|\frac{(-1+\alpha|\xi|^{1+\alpha})i}{(1+|\xi|^{1+\alpha})^2}\partial_\xi\hat{\varphi}\right\|_0 + \left\|\left(\frac{-i\xi}{1+|\xi|^{1+\alpha}}\right)\partial_\xi^2\hat{\varphi}\right\|_0$$

$$\leq c\left(\|\hat{\varphi}\|_0 + \left\|\partial_\xi \hat{\varphi}\right\|_0 + \left\|\partial_\xi^2 \hat{\varphi}\right\|_0\right) \leq c\|\varphi\|_{L^2_{2,0}}$$

Stein-weiss interpolation theorem (see theorem 5.4.1 in [1]) together with the previous inequality allows us to conclude $\|x^r A\varphi\|_0 \leq \|x^r\varphi\|_0$ for $0 \leq r \leq 2$.

For the case $2 < r < \frac{5}{2} + \alpha$, let $r = 2 + b$ with $0 < b < 1$ and $b < 1/2 + \alpha$.

$$\|A\varphi\|_{L^2_{r_1,0}} = \|x^{r_1} A\varphi\|_0$$

$$= \left\|D_\xi^b\left(\partial_\xi^2\left(\frac{-i\xi}{1+|\xi|^{1+\alpha}}\hat{\varphi}\right)\right)\right\|_0^2$$

$$\leq c\left\|D_\xi^b\left(\frac{|\xi|^{2\alpha+1}\mathrm{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right)\right\|_0 + c\left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right)\right\|_0 + c\left\|D_\xi^b\left(\frac{-1+\alpha|\xi|^{1+\alpha}}{(1+|\xi|^{1+\alpha})^2}\partial_\xi\hat{\varphi}\right)\right\|_0 + \left\|D_\xi^b\left(\frac{\xi}{1+|\xi|^{1+\alpha}}\partial_\xi^2\hat{\varphi}\right)\right\|_0$$

$$= A_1 + A_2 + A_3 + A_4$$

To bound A_2 we use the function χ defined in proposition 9

$$A_2 = \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right)\right\|_0 \tag{14}$$

$$\leq \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right)\right\|_0 + \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi)}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right)\right\|_0$$

$$\leq \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi}{(1+|\xi|^{1+\alpha})^3}\right)\cdot\hat{\varphi}\right\|_0 + \left\|\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi}{(1+|\xi|^{1+\alpha})^3}D_\xi^b\hat{\varphi}\right\|_0 + \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi)}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right)\right\|_0$$

$$\leq A_{2,1} + A_{2,2} + A_{2,3}$$

Let $g(\xi) = \frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)}{(1+|\xi|^{1+\alpha})^3}$, as a consequence of the proposition 7 and proposition 9 we get $D_\xi^b g(\xi) \in L^2(\mathbb{R})$. It is important to note that to use proposition 9, we must impose the condition that $b < 1/2 + \alpha$.

$$\left\|D_\xi^b(g(\xi))\right\|_0 = \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)}{(1+|\xi|^{1+\alpha})^3}\right)\right\|_0$$

$$\leq c\left(\||\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)\|_0 + \left\|\mathcal{D}_\xi^b(|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi))\right\|_0\right)$$

$$\leq K \quad \text{if} \quad b < \alpha + 1/2.$$

To bound $A_{2,1}$

$$(A_{2,1})^2 = \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)}{(1+|\xi|^{1+\alpha})^3}\right)\hat{\varphi}\right\|_0^2 \tag{15}$$

$$= \left\|D_\xi^b(g(\xi))\hat{\varphi}\right\|_0^2 \tag{16}$$

$$\leq \|\hat{\varphi}\|_\infty^2\left\|D_\xi^b(g(\xi))\right\|_0^2$$

$$\leq K^2\|\hat{\varphi}\|_{L^1}^2$$

$$\leq K^{1/2}\left(\int_{\mathbb{R}}|(1+x^2)^{r/2}\hat{\varphi}(x)\cdot\frac{1}{(1+x^2)^{r/2}}|dx\right)^2$$

$$\leq K^{1/2}\left(\left(\int_{\mathbb{R}}|(1+x^2)^{r/2}\hat{\varphi}(x)|^2dx\right)^{1/2}\left(\int_{\mathbb{R}}\frac{1}{(1+x^2)^r}dx\right)^{1/2}\right)^2$$

$$\leq c\|\varphi\|_0^2 + c\|\varphi\|_{L_r^2}^2.$$

The bound for the term $A_{2,2}$ is immediate and the bound for the term $A_{2,3}$ is easier, because we are on the set where the function $(1-\chi(\xi)) \neq 0$, so

$$A_{2,3} = \left\|D_\xi^b\left(\frac{|\xi|^\alpha \mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}\hat{\varphi}\right)\right\|_0$$

$$\leq \left\|D_\xi^b\left(\frac{|\xi|^\alpha \mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}\right)\hat{\varphi}\right\|_0 + \left\|\frac{|\xi|^\alpha \mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}D_\xi^b\hat{\varphi}\right\|_0$$

$$\leq \left\|D_\xi^b\left(\frac{|\xi|^\alpha \mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}\right)\right\|_\infty \|\hat{\varphi}\|_0 + \left\|\frac{|\xi|^\alpha \mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}\right\|_\infty \left\|D_\xi^b\hat{\varphi}\right\|_0$$

$$\leq c\|\varphi\|_0 + c\|\varphi\|_{L^2_{r_{1,0}}}$$

The bound for A_1 is obtained in a similar way as the bound for A_2, with the difference that the bound condition is $b < (2\alpha+1)+1/2$, which is true for all $0<\alpha<1$. The bounds for the terms A_3 and A_4 are a direct consequence of the proposition 7.

Now we need to bound the group $E(t)\varphi = \left(e^{\frac{-i\xi}{1+|\xi|^{1+\alpha}}t}\hat{\varphi}\right)^\vee = (F(\xi,t)\hat{\varphi})^\vee$ on weighted spaces $\mathcal{F}_{sr}$.

For this purpose it is necessary to calculate the partial derivatives of $F(\xi,t)$:

$$\partial_\xi F(\xi,t) = \frac{-i(1-\alpha|\xi|^{1+\alpha})t}{(1+|\xi|^{1+\alpha})^2}F$$

$$\partial_\xi^2 F(\xi,t) = \left(\frac{-i\alpha(1+\alpha)|\xi|^{2\alpha+1}\mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3} + \frac{i(1+\alpha)(2+\alpha)|\xi|^\alpha \mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3} + \frac{(1-\alpha|\xi|^{1+\alpha})^2t^2}{(1+|\xi|^{1+\alpha})^4}\right)F$$

$$\partial_\xi^3 F(\xi,t) = \left(\frac{i\alpha(1+\alpha)(2+\alpha)|\xi|^{1+3\alpha}t}{(1+|\xi|^{1+\alpha})^4} - \frac{i(1+\alpha)(4\alpha^2+8\alpha+6)|\xi|^{2\alpha}t}{(1+|\xi|^{1+\alpha})^4} + \frac{i\alpha(1+\alpha)(2+\alpha)|\xi|^{\alpha-1}t}{(1+|\xi|^{1+\alpha})^4} - \frac{\alpha(1+\alpha)(1-\alpha|\xi|^{1+\alpha})|\xi|^{2\alpha+1}sgn(\xi)t^2}{(1+|\xi|^{1+\alpha})^5}\right.$$

$$\left. + \frac{(1+\alpha)(2+\alpha)(1-\alpha|\xi|^{1+\alpha})|\xi|^\alpha \mathrm{sgn}(\xi)t^2}{(1+|\xi|^{1+\alpha})^5} - \frac{2\alpha(1+\alpha)(1-\alpha|\xi|^{1+\alpha})|\xi|^\alpha \mathrm{sgn}(\xi)t^2}{(1+|\xi|^{1+\alpha})^4} - \frac{4(1+\alpha)(1-\alpha|\xi|^{1+\alpha})^2|\xi|^\alpha \mathrm{sgn}(\xi)t^2}{(1+|\xi|^{1+\alpha})^5} + \frac{i(1-\alpha|\xi|^{1+\alpha})^3t^3}{(1+|\xi|^{1+\alpha})^6}\right)F$$

The following proposition bounds the group $E(t)$ for integer index.

Proposition 19: Let $E(t) = \left(e^{\frac{-i\xi}{1+|\xi|^{1+\alpha}}t}\hat{\varphi}\right)^\vee$. If $r=1$ or $r=2$ then

$$||E(t)\varphi||_{\mathcal{F}_{s,r}} \leq P_r(t)||\varphi||_{\mathcal{F}_{s,r}}$$

where $P_r(t)$ is a polynomial of degree r.

Proof. We show the case $r=2$. The case $r=1$ is similar.

$$\|E(t)\varphi\|_{\mathcal{F}_{s,2}} = \|E(t)\varphi\|_s + \|E(t)\varphi\|_{L_2^2}$$

$$\leq \|\varphi\|_s + \|x^2E(t)\varphi)\|_0$$

$$\leq \|\varphi\|_s + \left\|\partial_\xi^2(F\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + \left\|\frac{i\alpha(1+\alpha)|\xi|^{2\alpha+1}\text{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi}\right\|_0$$

$$+ \left\|\frac{i(1+\alpha)(2+\alpha)|\xi|^{\alpha}\text{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi}\right\|_0 + \left\|\frac{(1-\alpha|\xi|^{1+\alpha})^2 t^2}{(1+|\xi|^{1+\alpha})^4} F\hat{\varphi}\right\|_0$$

$$+ \left\|\frac{i(1-\alpha|\xi|^{1+\alpha})t}{(1+|\xi|^{1+\alpha})^2} F\,\partial_\xi\hat{\varphi}\right\|_0 + \left\|\partial_\xi^2\hat{\varphi}\right\|_0$$

$$\leq c\left(\|\varphi\|_s + t\|\hat{\varphi}\|_0 + t^2\|\hat{\varphi}\|_0 + t\left\|\partial_\xi\hat{\varphi}\right\|_0 + \left\|\partial_\xi^2\hat{\varphi}\right\|_0\right)$$

$$\leq P_r(t)\|\varphi\|_{\mathcal{F}_{s,2}}$$

The following proposition bounds the group $E(t)$ for non-integer index. We use the Stein derivative defined in 3.

Proposition 20: Let $E(t) = \left(e^{\frac{-i\xi}{1+|\xi|^{1+\alpha}}t}\hat{\varphi}\right)^{\vee}$, $0 \leq r < 5/2 + \alpha < 3$.

Then,

$$\| E(t)\varphi \|_{\mathcal{F}_{s,r}} \leq C(t) \| \varphi \|_{\mathcal{F}_{s,r}},$$

where $C(t)$ is a continuous increasing function in t.

Proof.

$$\| E(t)\varphi \|_{\mathcal{F}_{s,r}} = \| E(t)\varphi \|_s + \| E(t)\varphi \|_{L_r^2}$$

$$\leq \| \varphi \|_s + \| |x|^r E(t)\varphi \|_0$$

Let's suppose $r = b$ with $0 < b < 1$, using the properties of Stein derivative and lemma 8, we get:

$$\| E(t)\varphi \|_{\mathcal{F}_{s,r}} = \| \varphi \|_s + \| |x|^b E(t)\varphi \|_0$$

$$= \|\varphi\|_s + \left\|\mathcal{D}_\xi^b(F\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + \left\|\mathcal{D}_\xi^b F \cdot \hat{\varphi}\right\|_0 + \left\|F \cdot \mathcal{D}_\xi^b\hat{\varphi}\right\|_0$$

$$+\|c(\alpha,b)t^b\hat{\varphi}\|_0 + \left\|\mathcal{D}_\xi^b\hat{\varphi}\right\|_0$$

$$\leq C(t) \| \varphi \|_{\mathcal{F}_{s,r}}$$

now, let $r = 1 + b$ with $0 < b < 1$, using the proposition 7 we have

$$\| E(t)\varphi \|_{\mathcal{F}_{s,r}} = \| \varphi \|_s + \| |x|^{1+b} E(t)\varphi \|_0$$

$$= \|\varphi\|_s + \left\|D_\xi^b\,\partial_\xi(F\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + \left\|D_\xi^b(\partial_\xi F \cdot \hat{\varphi})\right\|_0 + \left\|D_\xi^b(F \cdot \partial_\xi\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + \left\|D_\xi^b\left(\frac{-i(1+\alpha|\xi|^{1+\alpha})t}{(1+|\xi|^{1+\alpha})^2}F \cdot \hat{\varphi}\right)\right\|_0 + \left\|\mathcal{D}_\xi^b F \cdot \partial_\xi\hat{\varphi}\right\|_0 + \|F\|_\infty\left\|\mathcal{D}_\xi^b(\partial_\xi\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + ct\left(\|F\hat{\varphi}\|_0 + \left\|\mathcal{D}_\xi^b(F\hat{\varphi})\right\|_0\right) + \left\|c(b)t^b\,\partial_\xi\hat{\varphi}\right\|_0 + \left\|\mathcal{D}_\xi^b(\partial_\xi\hat{\varphi})\right\|_0 \tag{17}$$

$$\leq c(t)\left(\|\varphi\|_s + \left\|F\mathcal{D}_\xi^b\hat{\varphi}\right\|_0 + \left\|\mathcal{D}_\xi^b F\hat{\varphi}\right\|_0 + \left\|\partial_\xi\hat{\varphi}\right\|_0 + \left\|\partial_\xi\hat{\varphi}\right\|_0 + \left\|\mathcal{D}_\xi^b\,\partial_\xi\hat{\varphi}\right\|_0\right)$$

$$\leq c(t)(\|\varphi\|_s + \||x|^b\varphi\|_0 + \||x|\varphi\|_0 + \||x|^{1+b}\varphi\|_0)$$

$$\leq c(t)\parallel\varphi\parallel_{\mathcal{F}_{s,r}}$$

Now let's suppose that $r = 2 + b$, with $0 < b < 1$ and $b < 1/2 + \alpha$

$$\parallel E(t)\varphi\parallel_{\mathcal{F}_{s,r}} = \parallel\varphi\parallel_s + \parallel |x|^{2+b}E(t)\varphi\parallel_0$$

$$\leq \|\varphi\|_s + \left\|D_\xi^b\,\partial_\xi^2(\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + \left\|D_\xi^b(\partial_\xi^2 F\cdot\hat{\varphi})\right\|_0 + \left\|D_\xi^b(\partial_\xi F\cdot\partial_\xi\hat{\varphi})\right\|_0 + \left\|D_\xi^b(F\cdot\partial_\xi^2\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + \left\|D_\xi^b\left(\frac{|\xi|^{2\alpha+1}\mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3}F\hat{\varphi}\right)\right\|_0 + \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3}F\hat{\varphi}\right)\right\|_0$$

$$+\left\|D_\xi^b\left(\frac{(1-\alpha|\xi|^{1+\alpha})^2t^2}{(1+|\xi|^{1+\alpha})^4}F\hat{\varphi}\right)\right\|_0 + \left\|D_\xi^b\left(\frac{(1-\alpha|\xi|^{1+\alpha})t}{(1+|\xi|^{1+\alpha})^2}F\,\partial_\xi\hat{\varphi}\right)\right\|_0$$

$$+\left\|D_\xi^b(F\cdot\partial_\xi^2\hat{\varphi})\right\|_0$$

$$\leq \|\varphi\|_s + B_1 + B_2 + B_3 + B_4 + B_5 \tag{18}$$

First lets bound B_2, using the function $\chi(\xi)$ defined in proposition 9:

$$B_2 = \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3}F\hat{\varphi}\right)\right\|_0$$

$$\leq \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)}{(1+|\xi|^{1+\alpha})^3}tF\hat{\varphi}\right)\right\|_0 + \left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}t\hat{\varphi}\right)\right\|_0$$

$$\leq t\left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)}{(1+|\xi|^{1+\alpha})^3}\right)\hat{\varphi}\right\|_0 + \left\|D_\xi^b(F\hat{\varphi})\right\|_0$$

$$+t\left\|D_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}F\hat{\varphi}\right)\right\|_0$$

$$\leq B_{2,1} + B_{2,2} + B_{2,3}.$$

The term $B_{2,1}$ was bounded on the equation 15 and the term $B_{2,2}$ was bounded on the equation 17, to bound $B_{2,3}$ we have in mind that we are on the set where the function $1 - \chi(\xi) \neq 0$, i.e,

$$B_{2,3} = \left\|\mathcal{D}_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}F\hat{\varphi}\right)\right\|_0 \tag{19}$$

$$\leq \left\|\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}\right\|_\infty \left\|\mathcal{D}_\xi^b(F\hat{\varphi})\right\|_0 + \left\|\mathcal{D}_\xi^b\left(\frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}\right)F\hat{\varphi}\right\|_0$$

$$\leq cB_{2,2} + c\|\hat{\varphi}\|_0.$$

B_1 is bounded in a similar way as B_2 but with the condition $b < (2\alpha + 1) + 1/2$, which is true for $0 < \alpha < 1$. The bounds for the terms B_3, B_4 and B_5 are obtained from the lemma 8 and the proposition 7.

Theorem 21: If $\varphi \in \mathcal{F}_{s,r}(\mathbb{R})$ with $s > 1/2$ and $0 \leq r < 5/2 + \alpha < 3$, then exists $T = T(\|\varphi\|_{\mathcal{F}_{s,r}}, M) > 0$ and an unique function $u \in C([0,T]; \mathcal{F}_{s,r})$ which satisfy the integral equation 9, also, the function $\varphi \to u$ associated to the equation 1 is continuous.

Proof. By propositions 18 and 20, where the operator A and the group $E(t)$ were bounded in $\mathcal{F}_{s,r}(\mathbb{R})$, and following the ideas of the theory on $H^s(\mathbb{R})$ we get the local well-posedness.

4.1. Unique continuation of solutions

Theorem 22: If $\varphi \in \mathcal{F}_{s,r}(\mathbb{R})$ with $s > 1/2$ and $0 \leq r \leq 5/2 + \alpha < 3$, where the conditions for well-posedness from theorem 21 are satisfied, let $u \in C([0,T]; \mathcal{F}_{s,r})$ the solution of the initial value problem 1, with $u(0) = \varphi$, such that $\int_{\mathbb{R}} \varphi(x)dx \geq 0$, if for two times $t_1 = 0 < t_2 < T$ we have $u(t_i) \in \mathcal{F}^{s_1,s_2}_{5/2+\alpha,0}$ $\quad j = 1,2$ then $u \equiv 0$.

Proof. Let $u \in C([0,T]; \mathcal{F}_{s,r}))$ the solution of 1. multiplying x^2 by the integral equation 9 we have,

$$x^2 u = x^2 E(t)\varphi + x^2 \int_0^t E(t-\tau)A(u^2(\tau))d\tau.$$

We will analyze the derivative $D_\xi^{1/2+\alpha}$ for $0 \leq \alpha < 1/2$ on both sides of equality. Applying the Fourier transform to the first term we get:

$$\partial_\xi^2(F\hat{\varphi}) = \partial_\xi^2 F \cdot \hat{\varphi} + 2\,\partial_\xi F \cdot \partial_\xi \hat{\varphi} + F \cdot \partial_\xi^2 \hat{\varphi}$$

$$= \frac{i\alpha(1+\alpha)|\xi|^{2\alpha+1}\mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi} + \frac{i(1+\alpha)(2+\alpha)|\xi|^{\alpha}\mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi} + \frac{(1-\alpha|\xi|^{1+\alpha})^2 t^2}{(1+|\xi|^{1+\alpha})^4} F\hat{\varphi} - \frac{2i(1-\alpha|\xi|^{1+\alpha})t}{(1+|\xi|^{1+\alpha})^2} F\,\partial_\xi\hat{\varphi} + F\,\partial_\xi^2\hat{\varphi}$$

$$= C_1 + C_2 + C_3 + C_4 + C_5$$

From the bounds for the terms $B_1,\ B_3,\ B_4$ and B_5 in (18), we have $D_\xi^{1/2+\alpha} C_i \in L^2(\mathbb{R})$ for $i = 1,3,4,5$. To bound the term C_2 we use the function χ from proposition 9:

$$C_2 = -\frac{i(1+\alpha)(2+\alpha)|\xi|^{\alpha}\mathrm{sgn}(\xi)t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi}$$

$$= \frac{K_\alpha|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi} + \frac{K_\alpha|\xi|^{\alpha}\mathrm{sgn}(\xi)(1-\chi(\xi))t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi}$$

$$= C_{2,1} + C_{2,2} \ \text{ where} K_\alpha = -i(1+\alpha)(2+\alpha)$$

Note that $D_\xi^{1/2+\alpha} C_{2,2} \in L^2(\mathbb{R})$ due to (19). To bound the term $C_{2,1}$, we rewrite it in the following way:

$$C_{2,1} = K_\alpha \frac{|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)t}{(1+|\xi|^{1+\alpha})^3} F\hat{\varphi}$$

$$= K_\alpha|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)t\hat{\varphi}\left(\frac{F}{(1+|\xi|^{1+\alpha})^3} - 1\right) + K_\alpha|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)t\hat{\varphi}$$

$$= D_1 + D_2.$$

Since $D_\xi^{1/2+\alpha} D_1 \in L^2(\mathbb{R})$ and

$$(x^2 E(t)\varphi)^\wedge = C_1 + D_1 + D_2 + C_{2,2} + C_3 + C_4 + C_5$$

then,

$$D_\xi^{1/2+\alpha}(x^2E(t)\varphi)^\wedge - D_2) \in L^2(\mathbb{R}),$$

that is,

$$D_\xi^{1/2+\alpha}((x^2E(t)\varphi)^\wedge - K_\alpha|\xi|^\alpha \mathrm{sgn}(\xi)\chi(\xi)t\hat{\varphi}(\xi)) \in L^2(\mathbb{R}). \quad (20)$$

For the term of the integral let $u^2 = v$ and the Plancherel theorem implies

$\|x^2E(t)A(v)\|_0 = \left\|\partial_\xi^2\left(F\frac{-i\xi}{1+|\xi|^{1+\alpha}}\hat{v}\right)\right\|_0$, thus

$$\partial_\xi^2\left(F\frac{-i\xi}{1+|\xi|}\hat{v}\right) = \frac{\alpha(1+\alpha)|\xi|^{2\alpha+2}t}{(1+|\xi|^{1+\alpha})^4}F\hat{v} - \frac{(1+\alpha)(2+\alpha)|\xi|^{1+\alpha}t}{(1+|\xi|^{1+\alpha})^4}F\hat{v}$$

$$+\frac{i(1-\alpha|\xi|^{1+\alpha})^2\xi t^2}{(1+|\xi|^{1+\alpha})^5}F\hat{v} + \frac{2(1-\alpha|\xi|^{1+\alpha})^2t}{(1+|\xi|^{1+\alpha})^4}F\hat{v} + \frac{2(1-\alpha|\xi|^{1+\alpha})\xi t}{(1+|\xi|^{1+\alpha})^3}F\,\partial_\xi\hat{v} + \frac{-i\alpha(1+\alpha)|\xi|^{2\alpha+1}\mathrm{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3}F\hat{v} + \frac{i(1+\alpha)(2+\alpha)|\xi|^\alpha\mathrm{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3}F\hat{v}$$

$$+\frac{-2i(1-\alpha|\xi|^{1+\alpha})}{(1+|\xi|^{1+\alpha})^2}F\,\partial_\xi\hat{v} + \frac{-i\xi}{1+|\xi|^{1+\alpha}}F\,\partial_\xi^2\hat{v}$$

$$= E_1 + E_2 + \cdots + E_9$$

$$D_\xi^{1/2+\alpha}E_i \in C([0,T];L^2(\mathbb{R})), \quad \text{for} \quad i = 2,3,\ldots,9 \quad i \neq 7.$$

Now we rewrite the term E_7:

$$E_7 = \frac{i(1+\alpha)(2+\alpha)|\xi|^\alpha\mathrm{sgn}(\xi)}{(1+|\xi|^{1+\alpha})^3}F\hat{v}$$

$$= -K_\alpha\frac{|\xi|^\alpha\mathrm{sgn}(\xi)(1-\chi(\xi))}{(1+|\xi|^{1+\alpha})^3}F\hat{v} - K_\alpha\frac{|\xi|^\alpha\mathrm{sgn}(\xi)\chi(\xi)}{(1+|\xi|^{1+\alpha})^3}F\hat{v}$$

$$= E_{7,1} - K_\alpha|\xi|^\alpha\mathrm{sgn}(\xi)\chi(\xi)\left(\frac{F}{(1+|\xi|^{1+\alpha})^3} - 1\right)\hat{v} - K_\alpha|\xi|^\alpha\mathrm{sgn}(\xi)\chi(\xi)\hat{v}$$

$$= E_{7,1} + E_{7,2} + E_{7,3}.$$

Since, $D_\xi^{1/2+\alpha}E_{7,1}, D_\xi^{1/2+\alpha}E_{7,2} \in C([0,T];L^2(\mathbb{R}))$ we have,

$$D_\xi^{1/2+\alpha}\left((x^2E(t)A(v))^\wedge - E_{7,3}\right) \in C([0,T];L^2(\mathbb{R})). \quad (21)$$

Note that 20 and 21, imply

$$D_\xi^{1/2+\alpha}\left((x^2u(t))^\wedge - D_2 - E_{7,3}\right) \in L^2(\mathbb{R})$$

for all $t \in [0,T]$. By hypothesis, exists t_2 such that $u(t_2) \in \mathcal{F}_{s,1/2+\alpha}$, then

$$D_\xi^{1/2+\alpha}\left(|\xi|^\alpha\mathrm{sgn}(\xi)\chi(\xi)\left(\int_0^{t_2}\hat{\varphi}(\xi) + \hat{v}(\tau,\xi)d\tau\right)\right) \in L^2(\mathbb{R}). \quad (22)$$

For convenience, we will write, $\hat{h}(\xi) = \left(\int_0^{t_2}\hat{\varphi}(\xi) + \hat{v}(\tau,\xi)d\tau\right)$, that transforms the previous identity into

$$D_\xi^{1/2+\alpha}(|\xi|^\alpha\mathrm{sgn}(\xi)\chi(\xi)\hat{h}(\xi)) \in L^2(\mathbb{R}),$$

which is the same as,

$$D_{\xi}^{1/2+\alpha}\big(|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)(\hat{h}(\xi)-\hat{h}(0))+|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)\hat{h}(0)\big)\in L^{2}(\mathbb{R})$$

Since,

$$D_{\xi}^{1/2+\alpha}\big(|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)(\hat{h}(\xi)-\hat{h}(0))\big)\in L^{2}(\mathbb{R}),$$

then,

$$D_{\xi}^{1/2+\alpha}\big(|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)\hat{h}(0)\big)\in L^{2}(\mathbb{R})$$

Theorem 9 states that $D_{\xi}^{b}(|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi))\in L^{2}(\mathbb{R})$ if and only if $b<1/2+\alpha$, then

$$D_{\xi}^{1/2+\alpha}\big(|\xi|^{\alpha}\mathrm{sgn}(\xi)\chi(\xi)\hat{h}(0)\big)\notin L^{2}(\mathbb{R}),$$

unless, $\hat{h}(0)=0$. This observation in our case becomes

$$\int_{0}^{t_2}\hat{\varphi}(0)+\hat{v}(\tau,0)d\tau=0,$$

that is,

$$\int_{0}^{t_2}\int_{\mathbb{R}}(\varphi(x)+v(\tau,x))dxd\tau=0.$$

Since $v=u^{2}$, we have

$$\int_{0}^{t_2}\int_{\mathbb{R}}(\varphi(x)+u^{2}(\tau,x))dxd\tau=0$$

This equality and the hypothesis, $\int_{\mathbb{R}}\varphi(x)dx\geq 0$ imply $u^{2}=0$ and therefore,$u\equiv 0$.

Note 23 The previous unique continuation principle was obtained under the hypothesis $0\leq\alpha<1/2$, as if $\alpha\geq 1/2$, we will have to bound $\partial_{\xi}^{3}E(t)$, but one of its terms is $\frac{i\alpha(1+\alpha)(2+\alpha)|\xi|^{\alpha-1}}{(1+|\xi|^{1+\alpha})^{4}}tF$, which has a singularity at zero. Thus, for $1/2\leq\alpha<1$, the persistence is obtained for $0\leq r<3$.

CONCLUSIONS

- Using Banach's fixed-point theorem we proved well-posedness in Sobolev spaces, for and weighted Sobolev spaces, for and.
- We proved global well-posedness in Sobolev spaces, for.
- The unique continuation principle was obtained, that is, if the initial data with and, and is the solution for the initial value problem (1), with and, $ and there also exists two times such that for then the solution is identically zero.

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